Formation of a New Class of Random Fractals in Fragmentation with Mass Loss

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Abstract

We consider the fragmentation process with mass loss and discuss self-similar properties of the arising structure both in time and space focusing on dimensional analysis. This exhibits a spectrum of mass exponents θ, whose exact numerical values are given for which $x^{-\theta}$ or $t^{\theta z}$ has the dimension of particle size distribution function $c(x,t)$ where z is the kinetic exponent. We also give explicit scaling solution for special case. Finally, we identify a new class of fractals ranging from random to non-random and show that the fractal dimension increases with increasing order and a transition to strictly self-similar pattern occurs when randomness is completely seized.

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The kinetics of irreversible and sequential breakup of particles occurs in a variety of physical processes and has important applications in science and technology. These include erosion [1], grinding and crushing of solids[2], polymer degradation and fiber length reduction [3], breakup of liquid droplets [4] etc. to name just a few. In recent years there has been an increasing interest in studying fragmentation allowing variations to increase the flexibility of the theory in matching the conditions of real phenomena such as extension to higher dimension [5], agglomerate erosion [1], mass loss [6], volume change [7], fragmentation-annihilation [8]. The kinetic equation approach of fragmentation is linear in character which makes it analytically tractable for a large class of breakup kernels. This is contrary to the reverse process, describing the kinetics of coagulation whose mean field approach proposed by Smoluchowski is non-linear in character and solved for a limited choice of collision kernels. This may reflect the fact that breaking up of objects follows less constraints than its reverse process. Despite its apparent simplicity and the fact that the first work appeared more than a century ago, the fragmentation process is still producing nontrivial results. For example, only recently it has been observed that when particles are described by more than one dynamical quantity, such as size and shape, the system exhibits multiscaling as it obeys infinitely many conservation laws [9]. Moreover, the resulting fragment distribution was shown to exhibit multifractality on a unique support when describing fragmentation and on one of infinitely many supports when describing stochastic Sierpinski gasket process [5]. It has also been discovered that a shattering transition occurs as the subsequent generation of fragments has a shorter life time than the fragments of previous generation [10]. McGrady and Ziff further showed in [10] that the shattering regime produces a fractal dust with dimension $0 < D_f < 1$ due to mass being lost to the phase of zero sized particles. In one dimension, there is only one phase boundary for shattering transition which is identified by the singularity of kinetic exponent whereas in more than one dimension there are multiple phase boundaries [5]. Shattering transition is also shown to be accompanied by the absence of scaling and self-averaging [11].

If associating disorder with broken objects is the most natural thing to do, then searching for an order even in this disorder is the next natural thing. This forms part of our motivation of this work. In this letter, we consider the kinetics of fragmentation with continuous mass loss and look more at the geometric and scaling aspects of the process than merely trying to solve the equation. The scaling theory essentially provides solutions in the long-time and short-size limit when the particle size distribution function evolves to a simpler form as well as becomes independent of initial conditions [12]. In reality, the most experimental system evolves to the point where this behaviour is reached. Our aim is to search for an order and quantify the arising geometry of the pattern. In fact, there are many physical processes that provide an intriguing

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connection between geometry and physics such as Percolation, Diffusion Limited Aggregation (DLA), Self-Organised Criticalitiy (SOC), etc. These processes evolve according to a random process obeying some conservation laws and creating simple geometrical structures that traditional Euclidean geometry cannot describe. Like many statistical physics problems, exact solutions for the distribution of masses of such ramified or stringy objects and their geometry vis-a-vis measuring their fractal dimension by complete analytical means, are still a challenge even in one dimension.

The evolution of particle size distribution function $c(x,t)$ for fragmentation with mass loss is

$$\frac{\partial c(x,t)}{\partial t} = -c(x,t)\int_0^\infty F(y,x-y)dy + 2\int_x^\infty dyc(y,t)F(x,y-x) + \frac{\partial}{\partial x}(m(x)c(x,t)) \tag{1}$$

where $F(x,y)$ is the breakup kernel describing the rate at which a particle of size $(x+y)$ breaks into sizes x and y. Fragmentation is a process whereby cuts are equivalent of seeds being sown on the fragmenting objects, thus producing two new segments. This immediately creates two more new ends belonging to the two different, newly-created fragments; in doing so, fragments start loosing their masses immediately (as if seeds were growing on either sides uniformly) until they encounter another seed or become dust-like thereby stopping loosing their masses. Therefore the model we consider can also mimic nucleation and growth of gap in one dimension which have some relevance in Kolmogorov-Avrami-Johnson-Mehl (KAJM) nucleation and growth processes and space covering by growing rays [13].

We consider the breakup kernel to be $F(x,y) = (xy)^\beta(x+y)^{\lambda-1}$, for which the breakup rate $a(x) = \int_0^x F(y,x-y)dy = px^{2\beta+\lambda}$, where $p = \frac{(\Gamma(\beta+1))^2}{\Gamma(2\beta+2)}$. The first term on the right hand side of the equation (1) reveals that $x^{-(2\beta+\lambda)}$ bears the dimension of time and this put a strong constraint on the mass loss term. So, the dimensional consistency requires $m(x) = mx^{2\beta+\lambda+1}$, with m a positive real constant. This dimensional consistency has been ignored in all previous studies [6] and $\gamma < 2\beta+\lambda+1$ was identified as the recession regime and $\gamma > 2\beta+\lambda+1$ as the fragmentation regime assuming $m(x) \sim mx^\gamma$. Since x and t are inextricably intertwined via the dimensional consistency, any of the two can be taken to be an independent parameter when the other one is expressible in terms of this. If x is chosen to be the independent parameter then the spatial scaling *ansatz* is $c(x,t) \sim x^{-\theta}\Phi(t/t_0(x))$, where $t_0(x) = x^{-(2\beta+\lambda)}$. On the other hand, if t is taken to be the independent parameter then the temporal scaling *ansatz* is $c(x,t) \sim t^{\theta z}\phi(x/x_0(t))$, with $x_0(t) = t^{-\frac{1}{2\beta+\lambda}}$ and the kinetic exponent $z = \frac{1}{2\beta+\lambda}$. The parameters $t/t_0(x) = \xi$ and $x/x_0(t) = \eta$ are the dimensionless quantities and so are $\Phi(\xi)$ and $\phi(\eta)$. Consequently, θ takes the value for which $x^{-\theta}$ and $t^{\theta z}$ have the dimension of $c(x,t)$. Note that the spatial and temporal scaling solution are trivially connected via $\Phi(\eta^\gamma) = \eta^\theta\phi(\eta)$, where $\eta = xt^{\frac{1}{\gamma}}$. The mass exponent θ can only be found if the system follows some conservation laws. For example, for pure fragmentation ($m=0$) the mass or size of the system is a conserved quantity and gives $\theta = 2$. Defining the n^{th} moment $M_n(t) = \int_0^\infty x^n c(x,t)dx$ and combining it with the rate equation (1) for the present choice of $F(x,y)$ and $m(x)$ yields

$$\frac{dM_n(t)}{dt} = -[\frac{(\Gamma(\beta+1))^2}{\Gamma(2\beta+2)} - \frac{2\Gamma(\beta+1)\Gamma(n+\beta+1)}{\Gamma(n+2\beta+2)} + mn]M_{n+2\beta+\lambda}(t). \tag{2}$$

The interesting feature of the above equation is that for $m > 0$, there are infinitely many $n = D_f(\beta,m)$ values for which $M_{D_f(\beta,m)}(t)$s are conserved quantities. However, for $m = 0$, there is only one conserved quantity $M_1(t)$, i.e. size or mass of the system, and this does not depend on β. We can find the $D_f(\beta,m)$ value by searching for the positive and real root of the equation

$$\frac{(\Gamma(\beta+1))^2}{\Gamma(2\beta+2)} - \frac{2\Gamma(\beta+1)\Gamma(n+\beta+1)}{\Gamma(n+2\beta+2)} + mn = 0 \tag{3}$$

which is polynomial in n of degree determined by the β value. Substituting the temporal scaling *anstaz* into the definition of $M_n(t)$ gives $M_n(t) \sim t^{-(n-(\theta-1))z}\int_0^\infty \eta^n\phi(\eta)d\eta$ and demanding $M_{D_f}(\beta,m)$ be a conserved quantity immediately gives $\theta = (1+D_f(\beta,m))$, which clearly depends on β and m only if $m > 0$. Owing to the random nature of the process and due to the presence of mass loss term, it is clear that when the process continues *ad infinitum*, it creates a distribution of points (dust) along a line at an extreme late stage. This distribution of points will inevitably be different from any known set such as strictly self-similar Cantor set, Julia set, Koch curve [14], stochastic or random Cantor set [15]. To measure the size of the set created in the long time limit, we define a line segment $\delta = \frac{M_n(t)}{M_{n-1}(t)} \simeq t^{-\frac{1}{2\beta+\lambda}}$. We can count the number of such segments

needed to cover the set and in the limit $\delta \longrightarrow 0$ (i.e.$t \longrightarrow \infty$), the number $N(\delta)$ will simply measure the set and appear to scale as $N(\delta) \sim \delta^{-D_f(\beta,m)}$. The exponent $D_f(\beta,m)$ is known as the Hausdorff-Basicovitch dimension of the set or as the fractal dimension which is simply the real positive root of the equation (3). To get a physical picture of the role played by m, we set $\beta = 0$ for the time being for which the equation (3) becomes quadratic in n and the real positive root is $D_f(m) = -\frac{1}{2}(1+1/m) + \frac{1}{2}\sqrt{(1+1/m)^2 + 4/m}$ when the second root is $D = -(D_f(m)+1+1/m)$. Therefore, the exponent θ is also function of m. The expression for $D_f(m)$ reveals that as m value increases, the fractal dimension decreases very sharply and in the limit $m \longrightarrow \infty$, $D_f(m) \longrightarrow 0$. This means that as m increases the size of the corresponding arising set decreases sharply due to fast disappearance of its member. Whereas, as $m \longrightarrow 0$, $D_f(m) \longrightarrow 1$, that is we recover the full set (pure fragmentation) that describes a line. On the other hand had we kept m fixed and let p decreases the effect would have been the same as we observed for increasing m with $p = 1$ (i.e. $\beta = 0$). Thus, it is the ratio between m and p that matters rather than their individual increases or decreases. To give a physical picture of what these results mean we define mass length relation for the object as $M_0 \sim \delta^{D_f(m)}$ and $M_e \sim \delta^d$ for the space where the object is being embedded, here d describes the Euclidean space. The density of the property of the object ρ then scales as

$$\rho \sim \delta^{D_f(m)-d}. \tag{4}$$

Note that for $m > 0$, $D_f(m)$ is always less than one. It is thus clear that for a given class of set created by a specific rule, when $D_f(m)$ decreases it means that it is increasingly moving away from d and hence more and more members from the full set are removed. This in turn creates increasingly ramified or stringy objects since $D_f(m) = d$ describes the compact object with uniform density. So, we show that increasing m/p ratio means that mass loss process gets stronger than the fragmentation process and vice versa.
We now attempt to find the spatial scaling solution for $\Phi(\xi)$. Note that the dimension of the arising pattern is independent of λ and consequently independent of how fast or slow the system performs the process. So, we can set $\lambda = 1$ without fear of missing any physics but it certainly simplifies our calculation. Substituting the spatial scaling *ansatz* into the rate equation (1) for $F(x,y) = 1$ and $m(x) = mx^2$ and differentiating it with respect to ξ, transforms the partial integro-differential equation into an ordinary differential equation ,

$$\xi(1-m\xi)\Phi''(\xi) + [(1-\theta) - \xi(2m(2-\theta)-1]\Phi'(\xi) - (m(2-\theta)(1-\theta) - (3-\theta))\Phi(\xi) = 0. \tag{5}$$

For $m = 1$ this is hypergeometric differential equation [16] whose only physically acceptable linearly independent solutions are ${}_2F_1(1, -(1+2D_f); -D_f; \xi)$ and $\xi^{(1+D_f)}\,{}_2F_1(2+D_f, -D_f; 2+D_f; \xi)$, where $D_f = 0.414213$. From these exact solutions for spatial scaling function we can obtain the asymptotic temporal scaling function $\phi(\xi) \sim e^{-D_f\xi}$ that satisfies the condition $\phi(\xi) \longrightarrow 0$ as $\xi \longrightarrow \infty$.
We now attempt to see the role of β on the system. To judge its role, it is clear from the previous discussion that we ought to give equal weight to all the terms in the equation (1) so that each of them can compete on an equal footing. This can be done if only we set $m = p = \frac{(\Gamma(\beta+1))^2}{\Gamma(2\beta+2)}$ so that the relative strength between fragmentation and mass loss process stays the same as β value increases. This is a very crucial point to be emphasized. We can obtain the fractal dimension for different values of β, which is simply the real positive root of the equation (3). A detailed survey reveals that the fractal dimension increases monotonically with increasing β. To find the fractal dimension in the limit $\beta \longrightarrow \infty$, we can use the Stirling's approximation in (3) to obtain $\ln[n+1] = (1-n)\ln[2]$ when $n = 0.4569997$ solves this equation. In order to give a physical picture of the role of β in the limit $\beta \longrightarrow \infty$, we consider the following model $F(x,y) = (x+y)^\gamma \delta(x-y)$. This model describes that cuts are only allowed to be in the middle in order to produce two fragments of equal size at each time event. This makes $a(x) = \frac{1}{2}x^\gamma$, so we need to choose $m(x) = \frac{1}{2}x^{\gamma+1}$, where $m = \frac{1}{2}$ gives the same weight as for the fragmentation process. Then the rate equation for $M_n(t)$ becomes

$$\frac{dM_n(t)}{dt} = -[\frac{(n+1)}{2} - 2^{-n}]M_{n+\gamma}(t). \tag{6}$$

As before we set the numerical factor of the right hand side of this equation equal to zero and then take natural log on both sides to obtain the n value for which $M_n(t)$ is time independent. In doing so, we arrive at the same functional equation for n as we found for $\beta \longrightarrow \infty$. This shows that the kernel $F(x,y) = (xy)^\beta(x+y)^{\lambda-1}$ behaves exactly in the same fashion as for $F(x,y) = (x+y)^\gamma\delta(x-y)$. We thus find that in the limit $\beta \longrightarrow \infty$, the resulting distribution of points is a set with fractal dimension $D_f = 0.4569997$ which is a strictly self-similar fractal as randomness is seized by dividing fragments into equal pieces. We are now in a position to

give a physical picture of the role played by β. First of all, the process with $\beta = 0$ describes the frequency curve of placing cuts about the size of the fragmenting particles is Poisson in nature. Consequently, the system enjoys the maximum randomness and the corresponding fractal dimension is $D_f = 0.414213$. Whereas, for $\beta > 0$, the frequency curve of placing cuts about the size of the fragmenting particles is Gaussian in nature meaning as β value increases particles are increasingly more likely to break in the middle than on either end. That is, as β increases, the variance decreases in such a manner that in the limit $\beta \longrightarrow \infty$ the variance of the frequency curve becomes infinitely narrow meaning a delta function distribution for which fragments are broken into two equal pieces. This analysis also specify that the rules determining the location where to place the cut are determined by the details of breakup kernel $F(x, y)$ rather than the breakup rate $a(x)$. So, there is a spectrum of fractal dimensions between $\beta \to 0$ when $D_f = 0.414213$ and $\beta \longrightarrow \infty$ when $D_f = 0.4569997$. A detailed numerical survey that we do not present here confirms that fractal dimension increases monotonically with β and reaches to a constant value when $\beta \longrightarrow \infty$ in a similar fashion as the variation of q with t during charging process in RC circuit. According to equation (4) increasing β vis-a-vis increasing order also means that the system losses less and less mass from the system and this happens despite the fact that now $\frac{m}{p}$ ratio stays the same. Perhaps it is note worthy to mention that the present model with $\beta = \lambda = 0$ and $m = 1$ correspond to Yule-Furry processes for cosmic shower theory with collision loss [17], though there too dimensional consistency was ignored.

In summary, we have identified a new set with a wide range of subsets produced by tuning the degree of randomness only. The process starts with an initiator of unit interval [0.1] and the generator divide the interval into two pieces and deleting some parts from either sides of both the pieces at each time step. The amount of the parts to be deleted is determined by the parameter that control the intensity of randomness. When this operation continues *ad infinitum*, what remains is an infinite number of dust scattered over the interval. We quantified the size of the arising set by fractal dimension and showed that the fractal dimension increases with increasing order and reaches its maximum value when the pattern described by the set is perfectly ordered, which is contrary to some recently found results [18]. We have also shown that the increase of fractal dimension and the increase of mass exponent θ go hand in hand since they are intimately connected. To the best of our knowledge the exact numerical value of this mass exponent has never been reported. We have given a scaling description of the process both in time and space and obtained explicit scaling function for special case of interest. Finally we argue on the basis of our findings that fractal dimension, degree of order and the extent of ramifications of the arising pattern are interconnected.

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